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Phase synchronization in time-delay systems

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Though the notion of phase synchronization has been well studied in chaotic dynamical systems without delay, it has not been realized yet in chaotic time-delay systems exhibiting non-phase-coherent hyperchaotic attractors. In this paper we report identification of phase synchronization in coupled time-delay systems exhibiting hyperchaotic attractor. We show that there is a transition from nonsynchronized behavior to phase and then to generalized synchronization as a function of coupling strength. These transitions are characterized by recurrence quantification analysis, by phase differences based on a transformation of the attractors, and also by the changes in the Lyapunov exponents. We have found these transitions in coupled piecewise linear and in Mackey-Glass time-delay systems.

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Synchronization is a natural phenomenon that one encounters in daily life. Since the identification of chaotic synchronization [1–3], several papers have appeared identifying and demonstrating basic kinds of synchronization both theoretically and experimentally (cf. [4,5]). Among them, chaotic phase synchronization (CPS) refers to the coincidence of characteristic time scales of the coupled systems, while their amplitudes of oscillations remain chaotic and often uncorrelated. Phase synchronization (PS) plays a crucial role in understanding the behavior of a large class of weakly interacting dynamical systems in diverse natural systems. Examples include circadian rhythm, cardiorespiratory systems, neural oscillators, population dynamics, electrical circuits, etc. [4–6].

The notion of CPS has been investigated so far in oscillators driven by external periodic force [7,8], chaotic oscillators with different natural frequencies and/or with parameter mismatches [9–12], arrays of coupled chaotic oscillators [13,14], and also in essentially different chaotic systems [15,16]. On the other hand PS in nonlinear time-delay systems, which form an important class of dynamical systems, have not yet been identified and addressed. A main problem here is to define even the notion of phase in time-delay systems due to the intrinsic multiple characteristic time scales in these systems. Studying PS in such chaotic time-delay systems is of considerable importance in many fields, as in understanding the behavior of nerve cells (neuroscience), where memory effects play a prominent role, in pathological and physiological studies, in ecology, in lasers, etc. [4–6,17–21].

In this paper, we report identification of phase synchronization in nonidentical time-delay systems in the hyperchaotic regime with non-phase-coherent attractors with unidirectional nonlinear coupling. We will show the entrainment of phases of a coupled piecewise linear time-delay system for weak coupling from the nonsynchronized state. Phase is calculated using the Poincaré method [4,5] after a transformation of attractors of the time-delay system, which looks then like a smeared limit cycle. The existence of PS and generalized synchronization (GS) in coupled time-delay systems is characterized by recently proposed methods based on recurrence quantification analysis and also in terms of Lyapunov exponents of the coupled time-delay systems.

We first consider the following unidirectionally coupled drive $x_1(t)$ and response $x_2(t)$ systems, which we have recently studied in detail in [22–24],

$$\dot{x}_1(t) = -ax_1(t) + b_1 f(x_1(t-\tau)), \tag{1a}$$

$$\dot{x}_2(t) = -ax_2(t) + b_2 f(x_2(t-\tau)) + b_3 f(x_1(t-\tau)), \tag{1b}$$

where b_1, b_2, and b_3 are constants, $a>0$, τ is the delay time, and $f(x)$ is a piecewise linear equation of the form

$$f(x) = \begin{cases} 0, & x \leqslant -4/3, \\ -1.5x-2, & -4/3 < x \leqslant -0.8, \\ x, & -0.8 < x \leqslant 0.8, \\ -1.5x+2, & 0.8 < x \leqslant 4/3, \\ 0, & x > 4/3. \end{cases} \tag{2}$$

We have chosen the values of parameters as $a=1.0$, $b_1=1.2$, $b_2=1.1$, and $\tau=15$, which are outside the region of complete, lag and anticipatory synchronizations discussed in [22,23]. For this parametric choice, in the absence of coupling, the drive $x_1(t)$ and the response $x_2(t)$ systems evolve independently. Further, in this case the drive $x_1(t)$ exhibits a hyperchaotic attractor [Fig. 1(a)] with five positive Lyapunov exponents (see [22] for the spectrum of Lyapunov exponents) and the response $x_2(t)$ has four positive Lyapunov exponents, i.e., both subsystems are qualitatively different (because $b_1 \neq b_2$). The parameter b_3 is the coupling strength of the unidirectional nonlinear coupling (1b), while the parameters b_1 and b_2 play the role of parameter mismatch resulting in nonidentical coupled time-delay systems.

Now the important questions we encounter are whether PS exists in the time-delay system (1) when the coupling is included ($b_3>0$) and, if so, how to characterize the possible transition to PS in such systems which possess in general highly non-phase-coherent attractors having a broadband

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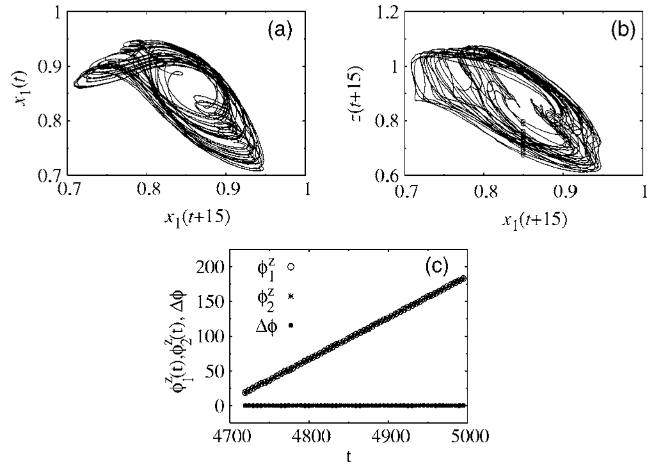

FIG. 1. Phase synchronization between the systems (1a) and (1b). (a) The non-phase-coherent hyperchaotic attractor of the uncoupled drive (1a). (b) Transformed attractor in the $x_1(t+\tau)$ and $z(t+\tau)$ space along with the Poincaré points represented as open circles. (c) The phases of the drive system (ϕ_1^z), the response system (ϕ_2^z) calculated from the new state variable $z(t+\tau)$, and their difference ($\Delta\phi$) for $b_3=1.5$.

power spectrum. In low-dimensional systems, a few methods [4,5] have been developed to define phase in phase-coherent chaotic attractors, which have a dominant peak in the power spectrum. The definition of phase is not so clear in noncoherent chaotic attractors, in particular in high-dimensional systems having broadband power spectra, such as time-delay systems. Methods to calculate the phase of noncoherent attractors of a time-delay systems are not readily available. One approach to calculate the phase of system (1) is based on the concept of curvature [25], which is often used in low-dimensional systems. However, we find that this procedure does not work in the case of time-delay systems in general, and in particular for Eqs. (1). We present here three other approaches to study PS in systems like as (1).

(i) We introduce a transformation to successfully capture the phase in the present problem. It transforms the non-phase-coherent attractor [Fig. 1(a)] into a smeared limit-cycle-like form with well-defined rotations around one center [Fig. 1(b)]. This transformation is performed by introducing the new state variable

$$z(t+\tau) = x_1(t)x_1(t+\hat{\tau})/x_1(t+\tau), \quad (3)$$

where $\hat{\tau}$ is the optimal value of delay time to be chosen (so as to rescale the original non-phase-coherent attractor into a smeared limit-cycle-like form). We plot the above attractor [Fig. 1(a)] in the $(x_1(t+\tau), z(t+\tau))$ phase space. The functional form of this transformation has been identified by generalizing the transformation used in the case of chaotic attractors in the Lorenz system [4]. We find the optimal value of $\hat{\tau}$ to be 1.65. The above transformation is obtained through a suitable functional form (along with a delay time $\hat{\tau}$), so as to unfold the original attractor [Fig. 1(a)] into a phase-coherent attractor. Now the attractor [Fig. 1(b)] looks indeed like a smeared limit cycle with nearly well-defined rotations around a fixed center. Hence, we can calculate the phase using the Poincaré method [4,5] from the rescaled attractor. The Poincaré points are shown as open circles in Fig. 1(b). The phases of both the drive $\phi_1^z(t)$ and the response $\phi_2^z(t)$ systems calculated from the state variable $z(t+\tau)$ are shown in Fig. 1(c) along with their phase difference $\Delta\phi=\phi_1^z(t)-\phi_2^z(t)$ for the value of the coupling strength $b_3=1.5$, showing a high-quality PS. This strong boundedness of the phase difference obtains for $b_3 \geqslant 1.382$. Note that the transformed attractor [Fig. 1(b)] does not have any closed loops as in the case of the original attractor [Fig. 1(a)]. If closed loops exist, they will lead to phase mismatch, and one cannot obtain exact matching of phases of both the drive and response systems as shown in Fig. 1(c).

(2) Next, we analyze the complex synchronization phenomena in the coupled time-delay systems (1) by means of very recently proposed methods based on recurrence plots [26]. These methods help to identify and quantify PS (particularly in non-phase-coherent attractors) and GS. For this purpose, the generalized autocorrelation function $P(t)$ [26] was introduced:

$$P(t) = \frac{1}{N-t}\sum_{i=1}^{N-t}\Theta(\epsilon - \|X_i - X_{i+t}\|), \quad (4)$$

where Θ is the Heaviside function, X_i is the ith data point of the system X, ϵ is a predefined threshold, $\|\cdot\|$ is the Euclidean norm, and N is the number of data points. Looking at the coincidence of the positions of the maxima of $P(t)$ for both systems, one can qualitatively identify PS.

A criterion to quantify PS is the cross-correlation coefficient between the drive, $P_1(t)$, and the response, $P_2(t)$, which can be defined as the correlation of probability of recurrence (CPR)

$$C_{\mathrm{CPR}} = \langle P_1(t)P_2(t)\rangle/\sigma_1\sigma_2, \quad (5)$$

where $\bar{P}_{1,2}$ means that the mean value has been subtracted and $\sigma_{1,2}$ are the standard deviations of $P_1(t)$ and $P_2(t)$ respectively. $C_{\mathrm{CPR}} \approx 1$ indicates that the systems are in complete PS, whereas for non-PS one obtains low values of CPR.

To characterize GS, the authors of [26] proposed the first index as the joint probability of recurrences (JPR),

$$J_{\mathrm{JPR}} = \frac{\frac{1}{N^2}\sum_{i,j}^{N}\Theta(\epsilon_x - \|X_i - X_j\|)\Theta(\epsilon_y - \|Y_i - Y_j\|) - R_{\mathrm{RR}}}{1 - R_{\mathrm{RR}}} \quad (6)$$

where R_{RR} is the rate of recurrence, and ϵ_x and ϵ_y are thresholds corresponding to the drive and response systems, respectively. The R_{RR} measures the density of recurrence points and it is fixed as 0.02 [26]. The JPR is close to 1 for systems in GS and is small when they are not in GS. The second index depends on the coincidence of probability of recurrence, which is defined as the similarity of probability of recurrence (SPR),

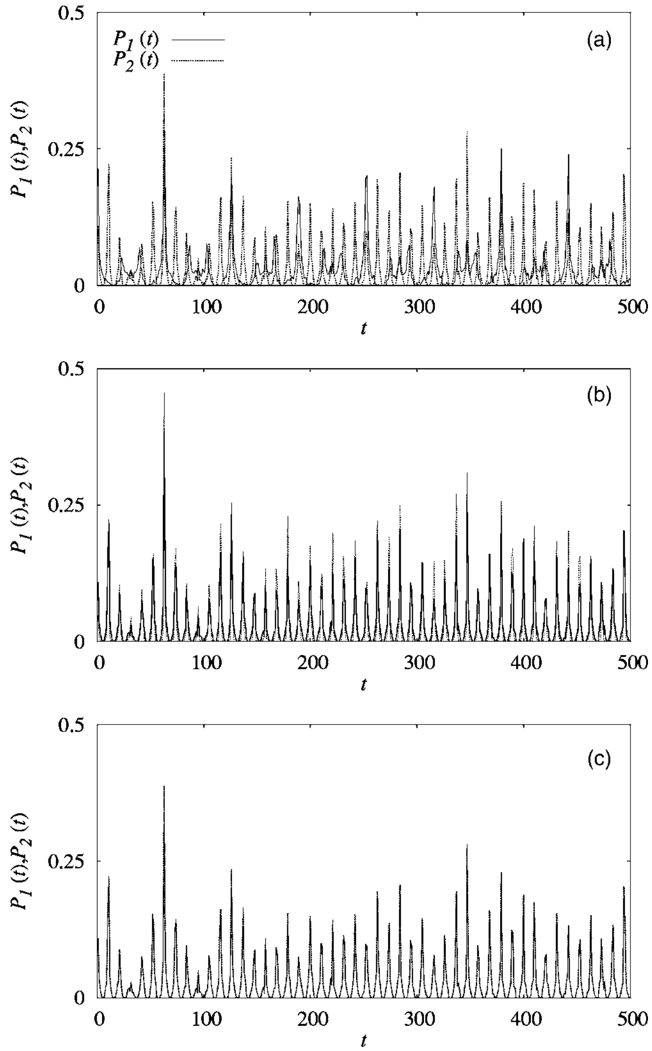

FIG. 2. Generalized autocorrelation functions of both the drive $P_1(t)$ and the response $P_2(t)$ systems. (a) Non-phase-synchronization for $b_3=0.6$, (b) phase synchronization for $b_3=1.5$, and (c) generalized synchronization for $b_3=2.3$.

$$S_{\mathrm{SPR}} = 1 - \langle (P_1(t) - P_2(t))^2 \rangle / \sigma_1 \sigma_2 . \quad (7)$$

The SPR is of order 1 if both the systems are in GS and approximately zero if they evolve independently.

Now, we will apply this concept to the original (nontransformed) attractor [Fig. 1(a)]. We estimate these recurrence-based measures from 5000 data points after sufficient transients with the integration step $h=0.01$ and sampling rate $\Delta t=100$. The generalized autocorrelation functions $P_1(t)$ and $P_2(t)$ [Fig. 2(a)] for the coupling $b_3=0.6$ show that the maxima of both systems do not occur simultaneously and there exists a drift between them, so there is no synchronization at all. This is also reflected in the rather low value of the CPR of 0.381. For $b_3 \in (0.91, 1.381)$, we observe the first substantial increase of recurrence reaching $C_{\mathrm{CPR}} \approx 0.5-0.6$. Looking at the details of the generalized correlation functions $P(t)$, we find that now the main oscillatory dynamics becomes locked, i.e., the main maxima of P_1 and P_2 coincide. For $b_3 \in (1.382, 2.2)$ the CPR reaches almost 1, i.e., now all maxima of P_1 and P_2 are in agreement and this is in accordance with strongly bounded phase differences. This is a strong indication for PS. Note, however, that the heights of the peaks are clearly different [Fig. 2(b)]. The differences in the peak heights exhibit that there is no strong interrelation in the amplitudes. Further increase of the coupling (here $b_3=2.3$) leads to the coincidence of both the positions and the heights of the peaks [Fig. 2(c)] referring to GS in systems (1). This is also confirmed from the maximal values of the indices $J_{\mathrm{JPR}}=1$ and $S_{\mathrm{SPR}}=1$, which is due to the strong correlation in the amplitudes of both systems. The transition from nonsynchronized to PS and then to GS is characterized by the maximal values of CPR, SPR, and JPR [Fig. 3(b)]. As expected from the construction of these functions, CPR refers mainly to the onset of PS, whereas JPR quantifies clearly the onset of GS. The existence of GS is also confirmed using the auxiliary system approach [27].

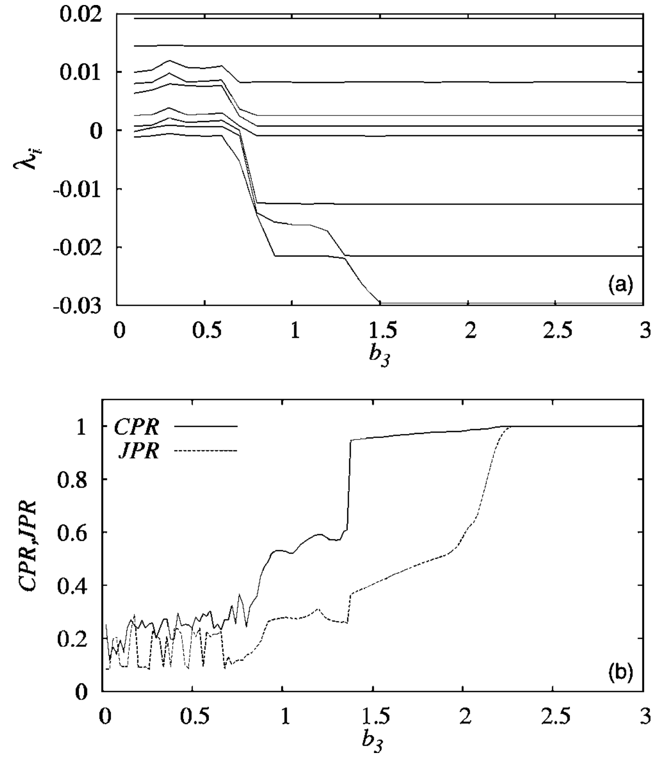

FIG. 3. (a) Spectrum of first nine largest Lyapunov exponents of the coupled systems (1) as a function of coupling strength $b_3 \in (0,3)$; (b) spectrum of CPR and JPR as a function of coupling strength $b_3 \in (0,3)$.

(3) The transition from nonsynchronization to PS is also characterized by changes in the Lyapunov exponents of the coupled time-delay systems (1). The spectrum of the nine largest Lyapunov exponents of the coupled systems is shown in Fig. 3(a), from which one can find that the Lyapunov exponents corresponding to the response system become negative from the value of the coupling strength $b_3 > 0.9$, where the transition to PS occurs, except for the largest Lyapunov exponent $\lambda_{max}^{(2)}$ which continues to remain positive. This is a strong indication that in this rather complex attractor the amplitudes become somewhat interrelated already at the transition to PS (as in the funnel attractor [25]). It is

interesting to note that the Lyapunov exponents of the response system λ_i (other than $\lambda_{max}^{(2)}$) are changing already at the early stage of PS ($b_3 \approx 0.91$), where the complete PS is not yet reached. This has been also observed for the onset of PS in phase-coherent oscillators [9].

We have obtained the same results for different sampling intervals Δt and for various values of delay time τ. We have also identified this transition to PS and to GS in the coupled Mackey-Glass systems [17,28], and also in (1) for linear coupling (unidirectional) using the above three approaches (these results will be presented elsewhere).

In conclusion, we have identified the existence of PS in coupled time-delay systems in the hyperchaotic regime with highly non-phase-coherent attractors. We have shown that there is a typical transition from a nonsynchronized state to PS for weak coupling and in the range of strong coupling there is a transition to GS from PS. We have also identified a suitable transformation, which works equally well for a Mackey-Glass system (having a more complex hyperchaotic attractor), to capture the phase of the underling attractor. We have also characterized the existence of PS and GS in terms of recurrence-based indices like the generalized autocorrelation function $P(t)$, CPR, JPR, and SPR and quantified the different synchronization regimes in terms of them. The above transition is also confirmed by the changes in the Lyapunov exponents. We have pointed out the existence of PS in coupled Mackey-Glass systems as well. The recurrence-based technique as well as the transformation used are also appropriate for the analysis of experimental data, i.e., we expect experimental verification of these findings.

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